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Robust model predictive control for norm-bounded uncertain systems using new parameter dependent terminal weighting matrix

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Abstract

A new robust model predictive control (MPC) technique is proposed for norm-bounded uncertain systems with input constraints. In order to improve feasibility and system performance, we propose an LMI condition for the cost monotonicity by using a new parameter dependent terminal weighting matrix. We formulate the problem as a minimization of the upper bound of infinite horizon cost function subject to the LMI condition for the cost monotonicity. A numerical example shows the effectiveness of the proposed method.

1. Introduction

During the past decade, the model predictive control (MPC) technique has received much attention due to its many advantages. MPC can easily handle constrained systems and time varying systems and provides good tracking performance. However, one of the drawbacks of MPC is its difficulty in incorporating plant model uncertainties. In order to solve this problem, Kothare et al. proposed a robust constrained MPC method for two types of uncertain system models [1]. One is a polytopic uncertain system model which is expressed by convex combination of different vertices of the uncertainty polytope and the other is a robust control model which is represented as a linear system with a feedback uncertainty. For stability of MPC, the terminal inequality was widely used with a terminal weighting matrix [5,6,7]. The terminal weighting matrix for the stability of the robust MPC should satisfy the terminal inequality condition over all admissible system uncertainties. Moreover, when we consider a robust MPC with input constraints, it is difficult to find a terminal weighting matrix satisfying such a condition for a wider range of uncertainties. Thus, Cuzzola et al. [2] improved system performance for the polytopic uncertain model by applying the parameter-dependent Lyapunov

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function (PDLF). Recently, Ding et al. [3] proposed a new robust MPC based on the technique of Cuzzola et al. [2]. The method enhanced feasibility and reduced the upper bound of the cost function by dividing infinite horizon control inputs into a set of free control inputs and linear state feedback control inputs in the terminal region. However, the robust MPC for the polytopic uncertain model has the problem of huge on-line computational burdens since the number of LMIs grows exponentially with the number of uncertainties and the prediction horizon N. Thus, Casavola et al. proposed a finite horizon MPC algorithm of norm-bounded uncertain systems for the first time [4]. But the basic procedure which involved LMI conditions derived off-line by means of the S-procedure. This can result in a performance degradation for an off-line design such that feasibility is increased. This leads to a trade-off between feasibility and performance.

In order to improve feasibility and performance at the same time, in this paper, we propose a new robust MPC technique for norm-bounded uncertain systems with input constraints. The proposed method has an on-line minimization algorithm to automatically resolve the trade-off. It is based on the minimization of the upper bound of a infinite horizon cost function subject to the cost monotonicity. A new LMI condition which meets cost monotonicity is derived by using a parameter dependent terminal weighting matrix. This condition also allow improvement of system performance by reducing the upper bound of worst-case cost functions with respect to uncertainties. The control inputs are obtained by solving the min–max problem, which is expressed in terms of linear matrix inequalities (LMIs) [8,9]. Finally, we demonstrate the effectiveness of the proposed approach using a numerical example.

In the sequel, the following notation will be used. $\mathbb{R}^n$ is the n-dimensional Euclidean space. $\mathbb{R}^{m\times n}$ denotes the set of $m\times n$ real matrix. $*$ denotes the symmetric part. $X>0$ ($X\geqslant 0$) means that X is a real symmetric positive definite matrix (positive semi-definite). I denotes the identity matrix with appropriate dimensions. diag$\{\cdots\}$ denotes the block diagonal matrix. $||x||^2_{\mathscr{W}}$ denotes $x^{\mathrm{T}}\mathscr{W}x$.

2. Problem statement and preliminaries

Consider a discrete-time robust control model with norm-bounded uncertainties:

$$\begin{aligned}&(k+1)=Ax(k)+Bp(k)+B_u u(k),\\&q(k)=Cx(k)+Dp(k),\\&p(k)=\Delta(k)q(k),\end{aligned}\tag{1}$$

where $x(k)\in\mathbb{R}^n$ is the state, $u(k)\in\mathbb{R}^{n_u}$ is the control input with constraints such as

$$-\overline{u}\leqslant u(k)\leqslant\overline{u},\quad \text{for all } k\in[0,\infty),\tag{2}$$

$p(k), q(k)\in\mathbb{R}^{n_p}$ are additional variables accounting for the uncertainty and $\Delta(k)\in\mathbb{R}^{n_p\times n_p}$ is a norm-bounded time-varying matrix in a set, defined as

$$\begin{aligned}&\varDelta=\{\varDelta(k)|\varDelta(k)=\mathrm{diag}[\delta_1(k)I,\delta_2(k)I,\ldots,\delta_p(k)I],\\&||\delta_i(k)||\leqslant 1, i=1,2,\ldots,p,\quad \text{for all } k\in[0,\infty)\},\end{aligned}\tag{3}$$

The goal in the paper is to design a stabilizing controller $u(k)$ for system (1) by model predictive control strategy. Now let $x(k+j|k)$ and $u(k+j|k)$ be predicted state variables and input variables, respectively.

In order to find such a controller, we consider the following min–max problem:

$$\underset{u(k)}{\text{Minimize}}\ \underset{\Delta(k+j),j\geqslant 0}{\text{Max}}\ J^{\infty}(k)\triangleq\sum_{j=0}^{N-1}\{||x(k+j|k)||^2_{\mathscr{Q}}+||u(k+j|k)||^2_{\mathscr{R}}\}+\sum_{j=N}^{\infty}\{||x(k+j|k)||^2_{\mathscr{Q}}+||u(k+j|k)||^2_{\mathscr{R}}\},\tag{4}$$

subject to

$$-\overline{u}\leqslant u(k+j)\leqslant\overline{u},\quad j\in[0,N-1],\tag{5}$$

$$-\overline{u}\leqslant u(k+j)=K(k)x(k+j|k)\leqslant\overline{u},\quad j\in[N,\infty),\tag{6}$$

for all $k\in[0,\infty)$, where $\mathscr{Q}$ and $\mathscr{R}$ are positive definite symmetric matrices.

In order to determine the gain matrix $K(k)$, we define a parameter dependent quadratic function

$$V(j,\Delta(k))=x(k+j|k)^{\mathrm{T}}P_f(\Delta(k+j))x(k+j|k),\quad j\geqslant N\tag{7}$$

subject to $P_f(\varDelta(k+j))=P_f^{\mathrm{T}}(\varDelta(k+j))>0$ and

$$\begin{aligned}\Delta V(j,\varDelta(k))&=x(k+j+1|k)^{\mathrm{T}}P_f(\varDelta(k+j+1))x(k+j+1|k)-x(k+j|k)^{\mathrm{T}}P_f(\varDelta(k+j))x(k+j|k)\\&<-||x(k+j|k)||^2_{\mathscr{Q}}-||K(k)x(k+j|k)||^2_{\mathscr{R}}.\end{aligned}\tag{8}$$

Summing (8) form $j = N$ to ∞ obtains an upper bound of the cost function (4)

$$J^{\infty}(k) \leqslant \sum_{j=0}^{N-1}\{||x(k+j|k)||^2_{\mathscr{Q}} + ||u(k+j|k)||^2_{\mathscr{R}}\} + ||x(k+N)||^2_{P_f(\Delta(k+N))}. \tag{9}$$

Thus, the min–max problem (4) is turned into the following min–max problem:

$$\underset{U(k),K(k),P_f(\Delta(k))}{\text{Minimize}} \quad \underset{\Delta(k+j),j\in[0,N-1]}{\text{Max}} J(k,k+N) \triangleq \sum_{j=0}^{N-1}\{||x(k+j|k)||^2_{\mathscr{Q}} + ||u(k+j|k)||^2_{\mathscr{R}}\} + ||x(k+N)||^2_{P_f(\Delta(k+N))}, \tag{10}$$

where $U(k) \triangleq [u^{\mathrm{T}}(k—k),\ldots,u^{\mathrm{T}}(k+N-1—k)]^{\mathrm{T}}$.

Since the cost function $J(k,k+N)$ is quadratic function with respect to the decision variable $U(k)$, we can solve the minimization problem (10) by using the following semi-definite programming:

$$\underset{\gamma_1(k),\gamma_2(k),U(k),K(k),P_f(\Delta(k))}{\text{Minimize}} \quad \gamma_1(k)+\gamma_2(k) \tag{11}$$

$$\text{subject to} \qquad J_1(k) \leqslant \gamma_1(k) \quad \text{and} \quad J_2(k) \leqslant \gamma_2(k) \quad \text{for all } \Delta(k+j) \in \mathbf{\Delta}, \quad j \in [0, N-1], \tag{12}$$

where $J_1(k) = \sum_{j=0}^{N-1}\{\|x(k+j|k)\|^2_{\mathscr{Q}} + \|u(k+j|k)\|^2_{\mathscr{R}}\}$, and $J_2(k) = \|x(k+j|k)\|^2_{\mathscr{P}_f(\Delta(k+N))}$.

Assume that the min–max problem (11) has the solutions $U^{*}(k)$, $K^{*}(k)$, $P^{*}_f(\Delta(k))$, associated with the minimum cost $J^{*}(k,k+N)$ at time k. Then, in order to satisfy the cost monotonicity, we need the following lemma.

Lemma 1. *If there exists* $P_f(\Delta(k+j)) > 0$ *and* $K(k)$ *satisfying*

$$\begin{aligned}\Delta V(j,k) &= x(k+j+1|k)^{\mathrm{T}}P_f(\Delta(k+j+1))x(k+j+1|k) - x(k+j|k)^{\mathrm{T}}P_f(\Delta(k+j))x(k+jk) \\ &< -||x(k+j|k)||^2_{\mathscr{Q}} - ||K(k)x(k+j|k)||^2_{\mathscr{R}},\end{aligned} \tag{13}$$

for any $\Delta(k) \in \mathbf{\Delta}$, $j \geqslant N$, then

$$J^{*}(k,k+N) > J^{*}(k+1,k+N+1). \tag{14}$$

Proof. From the definition of the cost function, we have

$$\begin{aligned}\Delta J^{*}(k) &= J^{*}(k+1,k+N+1) - J^{*}(k,k+N) \\ &= \sum_{j=0}^{N-1}\{||x(k+j+1|k+1)||^2_{\mathscr{Q}} + ||u^{*}(k+j+1|k+1)||^2_{\mathscr{R}}\} + ||x(k+N+1|k+1)||^2_{P^{*}_f(\Delta(k+N+1))} \\ &\quad - \sum_{j=0}^{N-1}\{||x(k+j|k)||^2_{\mathscr{Q}} + ||u^{*}(k+j|k)||^2_{\mathscr{R}}\} - ||x(k+N|k)||^2_{P^{*}_f(\Delta(k+N))}.\end{aligned} \tag{15}$$

If we use $U^{*}(k)$, $K^{*}(k)$, $P^{*}_f(\Delta(k+N))$ instead of $U^{*}(k+1)$, $K^{*}(k+1)$, $P^{*}_f(\Delta(k+N+1))$ at the time $k+1$, then by optimality

$$\begin{aligned}\Delta J^{*}(k) &\leqslant \sum_{j=0}^{N-1}\{||x(k+j+1|k+1)||^2_{\mathscr{Q}} + ||u^{*}(k+j+1|k)||^2_{\mathscr{R}}\} + ||x(k+N+1|k+1)||^2_{P^{*}_f(\Delta(k+N))} \\ &\quad - \sum_{j=0}^{N-1}\{||x(k+j|k)||^2_{\mathscr{Q}} + ||u^{*}(k+j|k)||^2_{\mathscr{R}}\} - ||x(k+N|k)||^2_{P^{*}_f(\Delta(k+N))},\end{aligned} \tag{16}$$

where $||x(k+j+1|k+1)||^2_{\mathscr{Q}} \leqslant ||x(k+j+1|k)||^2_{\mathscr{Q}}$, since $x(k+1—k+1)$ is measured state and $x(k+1—k)$ is the predicted state which is expressed by Eq. (1) for any uncertainties $\Delta(k) \in \mathbf{\Delta}$, $k \geqslant 0$.

Therefore, we have

$$\begin{aligned}\Delta J^{*}(k) &\leqslant \sum_{j=0}^{N-1}\{||x(k+j+1|k)||^2_{\mathscr{Q}} + ||u^{*}(k+j+1|k)||^2_{\mathscr{R}}\} + ||x(k+N+1|k)||^2_{P^{*}_f(\Delta(k+N))} \\ &\quad - \sum_{j=0}^{N-1}\{||x(k+j|k)||^2_{\mathscr{Q}} + ||u^{*}(k+j|k)||^2_{\mathscr{R}}\} - ||x(k+N|k)||^2_{P^{*}_f(\Delta(k+N))} \\ &= ||x(k+N|k)||^2_{\mathscr{Q}} + ||u^{*}(k+N|k)||^2_{\mathscr{R}} + ||x(k+N+1|k)||^2_{P^{*}_f(\Delta(k+N))} - ||x(k|k)||^2_{\mathscr{Q}} \\ &\quad - ||u^{*}(k|k)||^2_{\mathscr{R}} - ||x(k+N|k)||^2_{P^{*}_f(\Delta(k+N))}.\end{aligned} \tag{17}$$

Since the inequality (13) is satisfied for any $u^*(k+N|k) = K^*(k)x(k+N|k)$, it holds

$$\Delta J^*(k) < -\{||x(k|k)||^2_{\mathcal{Q}} + ||u^*(k|k)||^2_{\mathcal{R}}\}. \tag{18}$$

This completes the proof. □

3. Main results

In this section, we propose a new MPC technique to design a controller for the system (1) which improves feasibility and performance by deriving a new sufficient condition for the cost monotonicity. The minimization problem (11) is solved in two steps. First, we derive a new sufficient condition for the cost monotonicity using a new terminal weighting matrix. Then, we minimize the upper bound $\gamma_1(k) + \gamma_2(k)$ of the cost function $J(k, k+N)$, satisfying the sufficient condition. The parameter dependent terminal weighting matrix yield less conservative condition in terms of LMI's so that it allows to design a more robust MPC.

3.1. A new LMI condition for the cost monotonicity

For simplicity, let us define new variables.

$$s(k) = \begin{bmatrix} x(k) \\ p(k) \end{bmatrix}. \tag{19}$$

Then, for state feedback $u(k) = K(k)x(k)$, the state equation of the system (1) is rewritten as

$$s(k+1) = \begin{bmatrix} \overline{A} + \overline{B}_u\overline{K}(k) & \overline{B} \end{bmatrix} \begin{bmatrix} s(k) \\ p(k+1) \end{bmatrix}, \tag{20}$$

where

$$\overline{A} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \overline{B}_u = \begin{bmatrix} B_u \\ 0 \end{bmatrix}, \quad \overline{K}(k) = \begin{bmatrix} K(k) & 0 \end{bmatrix}. \tag{21}$$

Also, the variable $q(k+1)$ is expressed as follows:

$$q(k+1) = \begin{bmatrix} \bar{C} + \bar{C}_u\overline{K}(k) & D \end{bmatrix} \begin{bmatrix} s(k) \\ p(k+1) \end{bmatrix}, \tag{22}$$

where

$$\bar{C} = \begin{bmatrix} CA & CB \end{bmatrix}, \quad \bar{C}_u = \begin{bmatrix} CB_u \end{bmatrix}.$$

In the inequality (13), we choose a parameter dependent matrix $P_f(\Delta(k+j))$, $j \geqslant N$, as follows:

$$P_f(\Delta(k+j)) = \begin{bmatrix} I \\ \Delta_a(k+j) \end{bmatrix}^{\mathrm{T}} P_a(k) \begin{bmatrix} I \\ \Delta_a(k+j) \end{bmatrix}, \tag{23}$$

where $\Delta_a(k+j) = (I - \Delta(k+j)D)^{-1}\Delta(k+j)C$, $P_a(k) = P_a^{\mathrm{T}}(k) > 0$ for all $\Delta(k+j) \in \mathbf{\Delta}, j \geqslant N$ and define

$$Q_a(k) = P_a^{-1}(k) = \begin{bmatrix} Q_{11}(k) & Q_{12}(k) \\ Q_{21}(k) & Q_{22}(k) \end{bmatrix}. \tag{24}$$

By using the parameter dependent matrix $P_f(\Delta(k+j))$, we derive an LMI condition satisfying the terminal inequality (13) for any uncertainty $\Delta(k) \in \mathbf{\Delta}$ in the following theorem.

Theorem 1.

The terminal inequality (13) is satisfied for any $\Delta(k+j) \in \mathbf{\Delta}$, $j \geqslant N$, if there exist $G(k)$, $\overline{H}(k)$, $\Lambda(k) > 0$ and $Q_a(k) = Q_a(k)^{\mathrm{T}} > 0$, subject to

$$\begin{bmatrix} G(k) + G^{\mathrm{T}}(k) - Q_a(k) & * & * & * & * & * \\ 0 & \Lambda(k) & * & * & * & * \\ \overline{A}G(k) + \overline{B}_u\overline{H}(k) & \overline{B}\Lambda(k) & Q_a(k) & * & * & * \\ \overline{C}G(k) + \overline{C}_u\overline{H}(k) & D\Lambda(k) & 0 & \Lambda(k) & * & * \\ \mathcal{Q}_a^{1/2}G(k) & 0 & 0 & 0 & I & * \\ \mathcal{R}^{1/2}\overline{H}(k) & 0 & 0 & 0 & 0 & I \end{bmatrix} \geqslant 0, \tag{25}$$

where

$$\mathcal{Q}_a = \begin{bmatrix} \mathcal{Q} & 0 \\ 0 & 0 \end{bmatrix}, \quad G(k) = \begin{bmatrix} X(k) & 0 \\ Y(k) & Z(k) \end{bmatrix}, \quad \overline{H} = \overline{K}(k)G(k). \tag{26}$$

Proof. Since $p(k+j+1|k) = \Delta(k+j+1)q(k+j+1|k) = \Delta(k+j+1)\{C\ x(k+j+1|k) + \ Dp(k+j+1|k)\}$, it holds that

$$p(k+j+1|k) = \Delta_a(k+j+1)x(k+j+1|k). \tag{27}$$

Thus, the terminal inequality (13) with the parameter dependent matrix $P_f(\Delta(k+j))$ can be written as

$$\Delta V(j,k) = s(k+j+1|k)^{\mathrm{T}} P_a(k)s(k+j+1|k) - s(k+j|k)^{\mathrm{T}} P_a(k)s(k+j|k) \leqslant -s(k+j|k)^{\mathrm{T}} \overline{\mathcal{Q}}(k)s(k+j|k), \tag{28}$$

where

$$\overline{\mathcal{Q}}(k) = \begin{bmatrix} \mathcal{Q} + K^{\mathrm{T}}(k)\mathcal{R}K(k) & 0 \\ 0 & 0 \end{bmatrix}. \tag{29}$$

Then we derive the following inequality by using variable $s(k)$ defined in (20):

$$\begin{aligned}
&s(k+j+1|k)^{\mathrm{T}} P_a(k)s(k+j+1|k) - s(k+j|k)^{\mathrm{T}} P_a(k)s(k+j|k) \\
&\quad = \begin{bmatrix} s(k+j|k) \\ p(k+j+1|k) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \overline{A} + \overline{B}_u\overline{K}(k) & \overline{B} \end{bmatrix}^{\mathrm{T}} P_a(k) \begin{bmatrix} \overline{A} + \overline{B}_u\overline{K}(k) & \overline{B} \end{bmatrix} \begin{bmatrix} s(k+j|k) \\ p(k+j+1|k) \end{bmatrix} \\
&\qquad - \begin{bmatrix} s(k+j|k) \\ p(k+j+1|k) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} I \\ 0 \end{bmatrix} P_a(k) \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} s(k+j|k) \\ p(k+j+1|k) \end{bmatrix} \\
&\quad \leqslant - \begin{bmatrix} s(k+j|k) \\ p(k+j+1|k) \end{bmatrix}^{\mathrm{T}} \overline{\mathcal{Q}}_a(k) \begin{bmatrix} s(k+j|k) \\ p(k+j+1|k) \end{bmatrix},
\end{aligned} \tag{30}$$

where

$$\overline{\mathcal{Q}}_a(k) = \begin{bmatrix} \overline{\mathcal{Q}} & 0 \\ 0 & 0 \end{bmatrix}.$$

Also, we obtain a following inequality from the norm bounded uncertainty property:

$$\begin{aligned}
p(k+j+1|k)^{\mathrm{T}} p(k+j+1|k) &= (Cx(k+j+1|k) + Dp(k+j+1|k))^{\mathrm{T}} \Delta(k+j+1|k)^{\mathrm{T}} \\
&\quad \times \Delta(k+j+1|k)(Cx(k+j+1|k) + Dp(k+j+1|k)) \\
&= \begin{bmatrix} s(k+j|k) \\ p(k+j+1|k) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \overline{C} + \overline{C}_u\overline{K}(k) & D \end{bmatrix}^{\mathrm{T}} \Delta(k+j+1|k)^{\mathrm{T}} \\
&\quad \times \Delta(k+j+1|k) \begin{bmatrix} \overline{C} + \overline{C}_u\overline{K}(k) & D \end{bmatrix} \begin{bmatrix} s(k+j|k) \\ p(k+j+1|k) \end{bmatrix} \\
&\leqslant \begin{bmatrix} s(k+j|k) \\ p(k+j+1|k) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \overline{C} + \overline{C}_u\overline{K}(k) & D \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \overline{C} + \overline{C}_u\overline{K}(k) & D \end{bmatrix} \\
&\quad \times \begin{bmatrix} s(k+j|k) \\ p(k+j+1|k) \end{bmatrix}.
\end{aligned} \tag{31}$$

Let us define

$$\overline{A}_{cl}(k) = \overline{A} + \overline{B}_u\overline{K}(k), \overline{C}_{cl}(k) = \overline{C} + \overline{C}_u\overline{K}(k). \tag{32}$$

Then we derive the following inequality from (31) by using a positive diagonal matrix $\Lambda(k)$:

$$\begin{bmatrix} s(k+j|k) \\ p(k+j+1|k) \end{bmatrix} \begin{bmatrix} \overline{C}_{cl}(k)^{\mathrm{T}} \Lambda^{-1}(k)\overline{C}_{cl}(k) & \overline{C}_{cl}(k)^{\mathrm{T}} \Lambda^{-1}(k)D \\ D^{\mathrm{T}} \Lambda^{-1}(k)\overline{C}_{cl}(k) & -\Lambda^{-1}(k) + D^{\mathrm{T}} \Lambda^{-1}(k)D \end{bmatrix} \begin{bmatrix} s(k+j|k) \\ p(k+j+1|k) \end{bmatrix} \geqslant 0. \tag{33}$$

Performing the S-procedure [8] with (30) and (33), then we have,

$$\begin{bmatrix} \overline{A}_{cl}(k)^{\mathrm{T}}P_a(k)\overline{A}_{cl}(k) - P_a(k) + \overline{\mathcal{Q}}(k) & \overline{A}_{cl}(k)^{\mathrm{T}}P_a(k)\overline{B} \\ \overline{B}^{\mathrm{T}}P_a(k)\overline{A}_{cl}(k) & \overline{B}^{\mathrm{T}}P_a(k)\overline{B} - \Lambda^{-1}(k) \end{bmatrix} + \begin{bmatrix} \overline{C}_{cl}(k)^{\mathrm{T}} \\ D^{\mathrm{T}} \end{bmatrix} \Lambda^{-1}(k) \begin{bmatrix} \overline{C}_{cl}(k) & D \end{bmatrix} \leqslant 0. \tag{34}$$

By Shur complement [8], we derive that

$$\begin{bmatrix} P_a(k) - \overline{\mathcal{Q}}(k) & * & * & * \\ 0 & \Lambda^{-1}(k) & * & * \\ \overline{A}_{cl}(k) & \overline{B} & Q_a(k) & * \\ \overline{C}_{cl}(k) & D & 0 & \Lambda(k) \end{bmatrix} \geqslant 0. \tag{35}$$

Using the fact that the matrix $(G(k)^{\mathrm{T}} - Q_a(k))Q_a(k)^{-1}(G(k) - Q_a(k))$ is nonnegative definite in the technique of de Oliveira et al. [10], the inequality (35) is guaranteed if and only if the following inequality is satisfied:

$$\begin{bmatrix} G(k) + G(k) - Q_a(k) - G(k)^{\mathrm{T}}\overline{\mathcal{Q}}(k)G(k) & * & * & * \\ 0 & \Lambda^{-1}(k) & * & * \\ \overline{A}_{cl}(k)G(k) & \overline{B} & Q_a(k) & * \\ \overline{C}_{cl}(k)G(k) & D & 0 & \Lambda(k) \end{bmatrix} \geqslant 0. \tag{36}$$

By Schur complement [8] and congruence transformation with a matrix $\mathrm{diag}\{I, \Lambda(k), I, I\}$ in Eq. (36), we have the inequality (25). This completes the proof. □

3.2. Model predictive controller design

A solution of the min–max problem (4) is obtained by minimizing the upper bound $\gamma_1(k) + \gamma_2(k)$ of the cost function $J(k, k+N)$ satisfying the terminal inequality (13). In order to design the robust model predictive controller for the system (1), we transform the minimization problem (11) into LMI problem satisfying the new sufficient condition for cost monotonicity.

To formulate the predicted state equation, we define augmented vectors

$$\begin{aligned} &\mathcal{X}(k) \triangleq [x^{\mathrm{T}}(k|k)\cdots x^{\mathrm{T}}(k+N-1|k)]^{\mathrm{T}}, \\ &\mathcal{P}(k) \triangleq [p^{\mathrm{T}}(k|k)\cdots p^{\mathrm{T}}(k+N-1|k)]^{\mathrm{T}}, \\ &\mathcal{Q}(k) \triangleq [q^{\mathrm{T}}(k|k)\cdots q^{\mathrm{T}}(k+N-1|k)]^{\mathrm{T}}, \\ &\mathcal{X}_0(k) \triangleq [x^{\mathrm{T}}(k|k)0\ldots 0]^{\mathrm{T}}, \end{aligned} \tag{37}$$

and define matrices as

$$\begin{aligned} &\widehat{A} \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 \\ A & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & A & 0 \end{bmatrix}, \quad \widehat{B}_u \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 \\ B_u & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & B_u & 0 \end{bmatrix}, \\ &\widehat{B} \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 \\ B & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & B & 0 \end{bmatrix}, \quad \Phi_{k_2,k_1}(A) \triangleq A^{k_2-k_1}, \\ &\widetilde{B}_u \triangleq [\Phi_{N,0}(A)B_u\Phi_{N-1,0}(A)B_u\ldots B_u], \\ &\widetilde{B} \triangleq [\Phi_{N,0}(A)B\Phi_{N-1,0}(A)B\ldots B]. \end{aligned} \tag{38}$$

Then the predicted state $\mathcal{X}(k)$ and Nth predicted state $x(k+N|k)$ can be written as

$$\mathcal{X}(k) = \widehat{E}U(k) + \widehat{F}\mathcal{P}(k) + \widehat{H}_0(k), \tag{39}$$

$$x(k+N|k) = \Phi_{N,0}(A)x(k|k) + \widetilde{B}_uU(k) + \widetilde{B}\mathcal{P}(k), \tag{40}$$

where

$$\widehat{E}\triangleq[I-\widehat{A}]^{-1}\widehat{B}_u,\quad \widehat{F}\triangleq[I-\widehat{A}]^{-1}\widehat{B},\quad \widehat{H}_0(k)\triangleq[I-\widehat{A}]^{-1}\mathscr{X}_0(k).$$

Also, augmented variables $\mathscr{Q}(k)$ and $\mathscr{P}(k)$ are expressed as follows:

$$\mathscr{Q}(k)=\widehat{C}\mathscr{X}(k)+\widehat{D}\mathscr{P}(k), \tag{41}$$

$$\mathscr{P}(k)=\widehat{\Delta}(k)\mathscr{Q}(k), \tag{42}$$

where

$$\begin{aligned}&\widehat{C}\triangleq\mathrm{diag}(C,\ldots,C),\widehat{D}\triangleq\mathrm{diag}(D,\ldots,D),\\&\widehat{\Delta}\triangleq\mathrm{diag}(\Delta(k|k),\ldots,\Delta(k+N-1|k)).\end{aligned} \tag{43}$$

Using variables $\mathscr{X}(k), U(k)$ and Eq. (40), we rewrite the cost function $J(k,k+N)$ as

$$J(k,k+N)=J_1(k)+J_2(k), \tag{44}$$

where

$$J_1(k)=\mathscr{X}^{\mathrm{T}}(k)\widehat{\mathscr{Q}}\mathscr{X}(k)+U^{\mathrm{T}}(k)\widehat{\mathscr{R}}U(k), \tag{45}$$

$$J_2(k)=x^{\mathrm{T}}(k+N|k)P_f(\Delta(k+N))x(k+N|k), \tag{46}$$

$$\widehat{\mathscr{Q}}\triangleq\mathrm{diag}(\mathscr{Q},\mathscr{Q},\ldots,\mathscr{Q}),\quad \widehat{\mathscr{R}}\triangleq\mathrm{diag}(\mathscr{R},\mathscr{R},\ldots,\mathscr{R}).$$

Then the min–max problem (11) subject to the cost monotonicity (13) can be solved by a semi-definite programming after all constraints are converted to LMIs:

Theorem 1. *The robust stabilizing controller design problem that minimizes an upper bound of the performance index* (4) *subject to* (5), (6), (13) *can be solved by the following semi-definite programming*:

$$\underset{\gamma_1(k),\gamma_2(k),U(k),X(k),Y(k),Z(k),H(k),Q(k)}{\mathrm{Minimize}}\ \gamma_1(k)+\gamma_2(k) \tag{47}$$

subject to

$$\begin{bmatrix}-\gamma_1(k) & * & * & * & *\\ 0 & -\Lambda_1(k) & * & * & *\\ \Psi_1(k) & \widehat{C}\widehat{F}\Lambda_1(k) & -\Lambda_1(k) & * & *\\ \widehat{\mathscr{Q}}^{1/2}(\widehat{E}U(k)+\widehat{G}_0(k)) & \widehat{\mathscr{Q}}^{1/2}\widehat{F}\Lambda_1(k) & 0 & -I & *\\ \widehat{\mathscr{R}}^{1/2}U(k) & 0 & 0 & 0 & -I\end{bmatrix}\leqslant 0 \tag{48}$$

$$\begin{bmatrix}-I & * & * & * & * & * & *\\ 0 & -\Lambda_2(k) & * & * & * & * & *\\ 0 & 0 & -\Lambda_3(k) & * & * & * & *\\ \Psi_1(k) & (\widehat{C}\widehat{F})\Lambda_2(k) & 0 & -\Lambda_2(k) & * & * & *\\ C\Psi_2(k) & C\widetilde{B}\Lambda_2(k) & D\Lambda_3(k) & 0 & -\Lambda_3(k) & * & *\\ \Psi_2(k) & \widetilde{B}\Lambda_2(k) & 0 & 0 & 0 & -\overline{Q}_{11}(k) & *\\ 0 & 0 & \Lambda_3(k) & 0 & 0 & -\overline{Q}_{21}(k) & -\overline{Q}_{22}(k)\end{bmatrix}\leqslant 0 \tag{49}$$

$$-\begin{bmatrix}\bar{u}^{\mathrm{T}} & \ldots & \bar{u}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\leqslant U(k)\leqslant\begin{bmatrix}\bar{u}^{\mathrm{T}} & \ldots & \bar{u}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}} \tag{50}$$

$$\begin{bmatrix}G(k)+G^{\mathrm{T}}(k)-\overline{Q}(k) & * & * & * & * & *\\ 0 & \Lambda_4(k) & * & * & * & *\\ \overline{A}G(k)+\overline{B}_u\overline{H}(k) & \overline{B}\Lambda_4(k) & \overline{Q}(k) & * & * & *\\ \overline{C}G(k)+\overline{C}_u\overline{H}(k) & D\Lambda_4(k) & 0 & \Lambda_4(k) & * & *\\ \mathscr{Q}_a^{1/2}G(k) & 0 & 0 & 0 & I & *\\ \mathscr{R}^{1/2}\overline{H}(k) & 0 & 0 & 0 & 0 & I\end{bmatrix}\geqslant 0 \tag{51}$$

$$\begin{bmatrix} G(k) & \overline{H}(k) \\ \overline{H}^{\mathrm{T}}(k) & \overline{Q}(k) \end{bmatrix} \geqslant 0, \quad G_{ii}(k) \leqslant \overline{u}_i^2(k), \tag{52}$$

where $\Lambda_1(k)$, $\Lambda_2(k) \in R^{Nn_p \times Nn_p}$, $\Lambda_3(k)$ and $\Lambda_4(k) \in R^{n_p \times n_p}$ are positive-definite diagonal matrix and

$$\begin{aligned} &\Psi_1(k) \triangleq \widehat{C}\widehat{E}U(k) + \widehat{C}\widehat{G}_0(k), \quad \Psi_2(k) \triangleq \Phi_{N-1,0}(A)x(k|k) + \widetilde{B}_u U(k) \\ &\overline{Q}(k) \triangleq \gamma_2(k)P_a^{-1}(k), \quad \overline{H}(k) = \overline{K}(k)\overline{Q}(k), \overline{Q}_{ij}(k) \text{ are subblocks of } \overline{Q}(k). \end{aligned} \tag{53}$$

Proof. The first inequality $J_1(k) \leqslant \gamma_1(k)$ in (12) is satisfied with Eq. (39) and (45), if and only if

$$\begin{bmatrix} 1 \\ \mathscr{P}(k) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} \begin{bmatrix} 1 \\ \mathscr{P}(k) \end{bmatrix} \leqslant 0, \tag{54}$$

where

$$\begin{aligned} &E_{11} \triangleq -\gamma_1(k) + [\widehat{E}U(k) + \widehat{G}_0(k)]^{\mathrm{T}} \widehat{\mathscr{Q}} [\widehat{E}U(k) + \widehat{G}_0(k)] + U^{\mathrm{T}}(k)\widehat{\mathscr{R}}U(k), \\ &E_{12} = E_{21}^{\mathrm{T}} \triangleq [\widehat{E}(k)U(k) + \widehat{G}_0(k)]^{\mathrm{T}} \widehat{\mathscr{Q}}\widehat{F}, E_{22} \triangleq \widehat{F}^{\mathrm{T}}\widehat{\mathscr{Q}}\widehat{F}. \end{aligned} \tag{55}$$

The second inequality $J_2(k) \leqslant \gamma_2(k)$ in (12) is satisfied with Eq. (40) and (46), if and only if

$$\begin{bmatrix} 1 \\ \mathscr{P}(k) \\ p_N(k) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Psi_2^{\mathrm{T}}(k) & 0 \\ \widehat{B}^{\mathrm{T}}(k) & 0 \\ 0 & I \end{bmatrix} P_a(k) \begin{bmatrix} \Psi_2(k) & \widehat{B}(k) & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} 1 \\ \mathscr{P}(k) \\ p_N(k) \end{bmatrix} \leqslant \gamma_2(k). \tag{56}$$

Also, the following inequality can be derived from norm-bounded properties and Eqs. (39),(41) and (42):

$$\begin{aligned} \mathscr{P}^{\mathrm{T}}(k)\mathscr{P}(k) &= \{\widehat{C}\mathscr{X}(k) + \widehat{D}\mathscr{P}(k)\}^{\mathrm{T}} \widehat{\Delta}(k)^{\mathrm{T}} \widehat{\Delta}(k) \{\widehat{C}\mathscr{X}(k) + \widehat{D}\mathscr{P}(k)\} \\ &\leqslant [\widehat{C}\widehat{E}U(k) + (\widehat{C}\widehat{F} + \widehat{D})\mathscr{P}(k) + \widehat{C}\widehat{H}_0(k)]^{\mathrm{T}} \times [\widehat{C}\widehat{E}U(k) + (\widehat{C}\widehat{F} + \widehat{D})\mathscr{P}(k) + \widehat{C}\widehat{H}_0(k)], \end{aligned} \tag{57}$$

$$\begin{aligned} p_N^{\mathrm{T}}(k)p_N(k) &= \{Cx(k+N|k) + Dp_N(k)\}^{\mathrm{T}} \Delta(k+N)^{\mathrm{T}} \Delta(k+N) \{Cx(k+N|k) + Dp_N(k)\} \\ &\leqslant \{C\Psi_2(k) + C\widetilde{B}(k)\mathscr{P}(k) + Dp_N(k)\}^{\mathrm{T}} \{C\Psi_2(k) + C\widetilde{B}(k)\mathscr{P}(k) + Dp_N(k)\}. \end{aligned} \tag{58}$$

Thus, we can obtain the following LMI condition from (54) and(57) by the S-procedure [8]:

$$\begin{aligned} &\begin{bmatrix} 1 \\ \mathscr{P}(k) \end{bmatrix}^{\mathrm{T}} \left\{ \begin{bmatrix} -\gamma_1(k) + U(k)^{\mathrm{T}}\widehat{\mathscr{R}}U(k) & 0 \\ 0 & -\Lambda_1(k) \end{bmatrix} + \begin{bmatrix} (\widehat{E}U(k) + \widehat{G}_0(k))^{\mathrm{T}} \\ \widehat{F}^{\mathrm{T}} \end{bmatrix} \widehat{\mathscr{Q}} \begin{bmatrix} \widehat{E}U(k) + \widehat{G}_0(k) & \widehat{F} \end{bmatrix} \right. \\ &\left. \times \begin{bmatrix} (\widehat{C}\widehat{E}U(k) + \widehat{C}\widehat{G}_0(k))^{\mathrm{T}} \\ (\widehat{C}\widehat{F} + \widehat{D})^{\mathrm{T}} \end{bmatrix} \Lambda_1(k) \begin{bmatrix} (\widehat{C}\widehat{E}U(k) + \widehat{C}\widehat{G}_0(k))^{\mathrm{T}} \\ (\widehat{C}\widehat{F} + \widehat{D})^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \right\} \begin{bmatrix} 1 \\ \mathscr{P}(k) \end{bmatrix} \leqslant 0. \end{aligned} \tag{59}$$

By the Shur complement [8], the following condition is obtained from (59):

$$\begin{bmatrix} -\gamma_1(k) & * & * & * & * \\ 0 & -\Lambda_1(k) & * & * & * \\ \widehat{C}\widehat{E}U(k) + \widehat{C}\widehat{G}_0(k) & (\widehat{C}\widehat{F} + \widehat{D})\Lambda_1(k) & -\Lambda_1(k) & * & * \\ \widehat{\mathscr{Q}}^{1/2}(\widehat{E}U(k) + \widehat{G}_0(k)) & \widehat{\mathscr{Q}}^{1/2}\widehat{F}\Lambda_1(k) & 0 & -I & * \\ \widehat{\mathscr{R}}_2^{1/2}U(k) & 0 & 0 & 0 & -I \end{bmatrix} \leqslant 0. \tag{60}$$

In $J_2(k) \leqslant \gamma_2(k)$, by substituting $\overline{Q}(k) = \gamma_2(k)P_a^{-1}(k)$ and performing the S-procedure with Eq. (58), the following LMI condition is also derived:

$$\begin{bmatrix} 1 \\ \mathscr{P}(k) \\ p_N(k) \end{bmatrix}^{\mathrm{T}} \left\{ \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\Lambda_2(k) & 0 \\ 0 & 0 & -\Lambda_3(k) \end{bmatrix} + \begin{bmatrix} (\widehat{C}\widehat{E}U(k) + \widehat{C}\widehat{G}_0(k))^{\mathrm{T}} \\ \widehat{C}\widehat{F}^{\mathrm{T}} \\ 0 \end{bmatrix} \Lambda_2(k) \begin{bmatrix} (\widehat{C}\widehat{E}U(k) + \widehat{C}\widehat{G}_0(k))^{\mathrm{T}} \\ \widehat{C}\widehat{F}^{\mathrm{T}} \\ 0 \end{bmatrix}^{\mathrm{T}} \right.$$

Table 1
Simulation results

Methods	Ding et al. [3]	Casavola et al. [4]	Proposed method
$\bar{\alpha}_M$	117.8	279.6	306.7
γ^*	1091.1	1416.9	1049.4

$$+\begin{bmatrix} 0 \\ (C\Psi_2(k)+C\widetilde{B}(k))^{\mathrm{T}} \\ D^{\mathrm{T}} \end{bmatrix} \Lambda_3(k) \begin{bmatrix} 0 \\ C\Psi_2(k)+C\widetilde{B}(k) \\ D \end{bmatrix}^{\mathrm{T}} + \begin{bmatrix} \Psi_2^{\mathrm{T}}(k) & 0 \\ \widetilde{B}^{\mathrm{T}} & 0 \\ 0 & I \end{bmatrix} \overline{Q}^{-1}(k) \begin{bmatrix} \Psi_2(k) & \widetilde{B} & 0 \\ 0 & 0 & I \end{bmatrix} \Bigg\} \begin{bmatrix} 1 \\ \mathscr{P}(k) \\ p_N(k) \end{bmatrix} \leqslant 0. \quad (61)$$

By the Shur complement [8], the condition (49) is obtained from Eq. (61). The input constraint (5) is expressed as

$$-[\,\bar{u}^{\mathrm{T}} \quad \ldots \quad \bar{u}^{\mathrm{T}}\,]^{\mathrm{T}} \leqslant U(k) \leqslant [\,\bar{u}^{\mathrm{T}} \quad \ldots \quad \bar{u}^{\mathrm{T}}\,]^{\mathrm{T}}, \quad (62)$$

and the input constraint (6) is satisfied if

$$\begin{bmatrix} G(k) & H(k) \\ H^{\mathrm{T}}(k) & \overline{Q}(k) \end{bmatrix} \geqslant 0, \quad G_{ii}(k) \leqslant \bar{u}_i^2. \quad (63)$$

By substituting $\overline{Q}(k) = \gamma_2(k)P^{-1}(k)$, a sufficient condition is derived as (51). □

3.3. Numerical example

In order to show the effectiveness of the proposed robust MPC technique in this paper, we revisit the example handled in Ding et al. [3]. The system is described by

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \alpha(k) & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad u(k), \quad (64)$$

where $\alpha(k)$ is time-varying parameter within $\alpha \in [\alpha_m, \alpha_M]$. The initial condition is $x_0 = [2,2]^{\mathrm{T}}$, the weighting matrices $\mathscr{Q} = I$ and $\mathscr{R} = I$ and the input constraint $|u(k)| \leqslant 1$. The system (64) can be represented by an equivalent system (1) with the following matrices:

$$A = \begin{bmatrix} 1 & 0 \\ \alpha_{\mathrm{nom}} & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [\,\alpha_{\mathrm{dev}} \quad 0\,], \quad B_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad D = 0, \quad (65)$$

where $\alpha_{\mathrm{dev}} = (\alpha_M - \alpha_m)/2$, $\alpha_{\mathrm{nom}} = (\alpha_M + \alpha_m)/2$. We choose the horizon $N = 2$ and perform feasibility and performance test by comparing with previous results. Firstly, when we keep $\alpha_m = 0.5$ and vary α_M, we find the maximum feasible solution $\bar{\alpha}_M$. Secondly, we compute the upper bound of cost with the parameter $\alpha(k) = 1.5 + \sin(k)$ for each technique. Table 1 shows the simulation results, respectively.

One can see that our result is superior than the results of existing works.

4. Conclusions

In this paper, a new robust MPC technique is proposed for uncertain systems with input constraints. The controller is designed by solving the min–max problem with the finite horizon cost function subject to a new LMI condition for cost monotonicity. In order to improve feasibility of the problem and performance of the system, we use a new parameter dependent terminal weighting matrix to derive a novel criterion for the terminal inequality. The optimization problem is expressed by LMIs. The effectiveness of the proposed method is demonstrated by a numerical example.

References

[1] Kothare MV, Balakrishanan V, Morari M. Robust constrained model predictive control using linear matrix inequalities. Automatica 1996;32:1361–79.

[2] Cuzzola FA, Geromel JC, Morari M. An improved approach for constrained robust model predictive control. Automatica 2002;38:1183–9.
[3] Ding B, Xi Y, Li S. A synthesis approach of on-line constrained robust model predictive control. Automatica 2004;40:163–7.
[4] Casavola A, Famularo D, Franze G. Robust constrained predictive control of uncertain norm-bounded linear systems. Automatica 2004;40:1865–76.
[5] Kim KB. Implementation of stabilizing receding horizon controls. Automatica 2002;38:1705–11.
[6] Rawlings JB, Muske KR. The stability of constrained receding horizon control. IEEE Trans Autom Control 1993;38:1512–6.
[7] Lee JW, Kwon WH, Choi JH. On stability of constrained receding horizon control with finite terminal weighting matrix. Automatica 1998;34:1607–12.
[8] Boyd S, Ghaoui L, Feron E, Balakrishanan V. Linear matrix inequalities in system and control theory. Studies in applied mathematics, 15. Philadelphia, Pennsylvania: SIAM; 1994.
[9] Gahinet P, Nemirovski A, Laub AJ, Chilali M. LMI control toolbox. Natick, Massachusetts: The Mathworks; 1995.
[10] de Oliveira MC, Bernussou J, Geromel e JC. A new discrete-time robust stability condition. Syst Control Lett 1999;37:261–5.